Reconfigurable Quantum Anomalous Spin Hall Effect from Phonon dispersion in Weil 2D half materials: Group-VI Dichalcogenides and \( \text{PtCl}_3, \text{PdBr}_3, \text{RuCl}_3, \text{Ptl}_3 \)

Available online 8 October 2019: https://communities.acs.org/people/LiubovLokot/content

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Spin Hall insulators have connected with respect to dissipationless spin transport via a sample with the spin-orbit interactions (SOI) effects. The extremum rings of the valence band in tensile strained zincblende GaN quantum well as well as the two valley shape of the valence band in GaN nanotube grown along c axis were a collection have attracted our attention both for their fundamental studies and for their device applications. The Quantum Spin Hall effects are shown to be related with intraband transitions of bulk GaN. In the framework of the effective mass theories we have solved the Schrödinger equation if the topological insulator transformation is achieved. The exact solutions of the Schrödinger equations as well as Quantum Spin Hall effect of intraband transitions of bulk GaN are found. For the hexagonal symmetry of GaN the Effective Hamiltonian based on C_{6v} point symmetry group was found. In the article for Quantum Hall effect of intraband phototransitions of bulk GaN the expressions of Berry curvature as well as Hall conductivities have been found when the topological insulator transformation is achieved. Spin Hall insulators have connected with the Topological insulator (TI) as well as Exciton insulators. The dissipationless spin transport via the sample are shown to be related with Spin Hall insulator self-consistent solution of the Schrödinger equations for electrons and holes and the Poisson equations at the presence of spatially varying quantum well potential due to the piezoelectric effect and the local exchange-correlation potential in ZnO quantum well. In ZnO/Zn,MgO quantum well the electron-hole pairing leads to the exciton insulator states. An exciton insulator states with a gap 3.4 eV of ZnO quantum well and at 310 K temperature are predicted. If the electron and hole are separated, their energy is higher on 0.2 meV than if they are paired. The particle-hole pairing leads to the Cooper instability.
Abstract In the article we have found that the Weyl Hall semiconductor (WHS) states allow the topological phase transition happened between two quantum anomalous Hall (QAH) insulator phase with opposite Chern numbers and we have based on phonon dispersion of Weil 2D half semiconductor monolayers group like PtCl₃. The ground state of monolayer PtCl₃ allows the creation of the 2D WHS states with the pairs of fully polarized fermions at Weyl points which robust under spin-orbit coupling (SOC). The two QAH phases with ϵ = ±1 have edge channels propagating in opposite directions. Because the chirality of the edge channel is determined by the sign of Chern number. We have predicted the switching of the Quantum Anomalous Spin Hall states by rotating the magnetization vector on the basis of PtCl₃ monolayer and we have based on phonon dispersion of Weil 2D half semiconductor monolayers group like PtCl₃. Landau-Ginsburg-Devonshire theory of thin ferroelectric polar-active nanofilms in incommensurate phases and semiconductor heterostructures is presented. The self-consistent solutions of the Euler-Lagrange equation for the polarization vector and the Maxwell equations for light which propagates along Oz axis in thin ferroelectric polar-active nanofilms have been found. Quantized solutions of one-dimensional Maxwell equations for thin ferroelectric films in Incommensurate phase with space dispersion have been specified. The analytical solutions of the Maxwell wave equations as well as natural optical gyrotropy effects are found in Rb₂ZnBr₄ as well as K₂SeO₄ Incommensurate phases crystals connected with giant light velocity as well as via interaction with coherent phonon oscillations. In the framework of the superspace symmetry group theories the Maxwell wave equations are solved which are shown to be connected with the symmetry group of D₁₆²ₙ or isomorphic groups. In the paper the non-zero gyration g₃₃ and gyrotropic birefringence ε₁₂ tensors of K₂SeO₄ and Rb₂ZnBr₄ materials based on D₁₆²ₙ space symmetry group were found. The values of natural optical gyrotropy as well as Rashba spin splitting are shown to be specified like (k(0) ± k(2))² as displacements of two symmetrically allocated parabolas from Brillouin zone center. In the article the natural optical gyrotropy effects are shown to be found with light velocity like e = ħck/eV = 14.0798 eV but the corresponding wave vector was estimated to be k = 7.161 × 10⁵ cm⁻¹. The found strong natural optical gyrotropy has been based on available experimental data [Phys. Rev. B 38, 8075, (1988)].

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1 Contents

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2 Introduction

It is known [1] that vibrations with sine-type modulations can be induced by the impulsive stimulated Raman-scattering process resulting in sine-type oscillations. It explains the coherent oscillations observed in transparent compounds under pumping with a photon energy smaller than an optical gap [1]. In an opaque materials the abrupt modification of pump photons can trigger displaced motions of ions towards new coordinates in the excited state resulting in cosine-type oscillations [1].

The optical activity in Rb₂ZnBr₄, Rb₂ZnCl₄, K₃SeO₄, (NH₄)₃BeF₄, (N(CH₃)₄)₂MnCl₄ crystals has been measured along three directions in a temperature interval from 400K to 50K [2]. In the incommensurate phase already a nonvanishing element of the gyration tensor has been observed despite the fact that the average crystal structure has inversion symmetry [2]. In order to explain this phenomenological space dependent dielectric and gyration tensors being invariant with respect to the superspace group of Rb₂ZnBr₄, Rb₂ZnCl₄, K₃SeO₄, (NH₄)₂BeF₄, (N(CH₃)₄)₂MnCl₄ have been considered [2, 3, 4, 5, 6, 7, 8, 9]. Recently one has seen a growing interest in systems like modulated crystals with charge or spin density waves which can be considered as crystals with a distortion which is periodic in space or in space-time [3, 2, 6]. The Euclidean symmetry of these systems was not a three-dimensional space group but a four-dimensional superspace groups [3, 2, 4, 5, 6]. It is known that the spin-orbit coupling originates from gradients of the Coulomb potentials in the atomic cores. The spin-orbit coupling has been presented by Δ = ℏdψ,x/2(p, σx)ψ, y, where V is the microscopic crystal potential of graphene. In graphene we would have Δ = 0 if the basis functions |ψ, x⟩ and |ψ, y⟩ were made up on basis of pure π or (p, σ) orbitals. However spin-orbit coupling induces a mixing of the π or (p, σ) and σ or (p, σ) orbitals in graphene that contributes to Δ in second order of spin-orbit coupling. In the our case [10, 11, 12, 13, 14, 15, 16, 17, 18, 19] the strain mediates a coupling between intrinsic-spin and orbital dynamics and the lowest order contribution p₂σz(εxx + εyy)σxσz constitutes a renormalization of the intrinsic spin-orbit coupling p₂σzσxσz. In the article [20] a creation of giant spin-orbit splitting (~ 100 meV) of the graphene Dirac cone up to the Fermi energy are shown to be related with Au intercalation at the graphene-Ni interface. Photoelectron spectroscopy reveals the hybridization with Au 5d states as the source for this giant splitting. A sharp graphene-Au interface at the equilibrium distance accounts for only ~ 10 meV spin-orbit splitting and enhancement is due to the Au atoms in the hollow position that get closer to graphene and do not break the sublattice symmetry.

3 Tight-binding method of phonon modes dispersion in Weil 2-Dimensional half semiconductors: Group—VI Dichalcogenides and PtCl₃, PdBr₃, RuCl₃, PtI₃, Cr₂Ge₂Te₆, VCl₃, OsCl₃

The two-dimensional Group-VI Dichalcogenides include layered materials: MoX₂, WX₂, (X=S, Se, Te) [21]. The two-dimensional Weil half semiconductors include the next layered materials with the honeycomb lattice with two atoms per unit cell: PtCl₃, graphene, PtBr₃, Pt₃, PdCl₃, PdBr₃, PdI₃, RuCl₃, Cr₂Ge₂Te₆, VCl₃, OsCl₃, CrI₃, VSe₂ [22, 23, 24]. For explanation of strong spin-orbit splitting and enhancement due to the Au atoms in the hollow position that get closer to graphene and do not break the sublattice symmetry we have calculated phonon dispersion of graphene as well as
their wave vectors with accounts oscillate interactions [25, 26, 11, 27, 28, 29, 30, 21, 31, 32] between the first and second nearest neighbors in the basis of the Born-von Karman model [20, 24, 1, 9, 10, 33, 34, 25, 11, 27, 21, 12, 16, 17]. Analytical expressions were derived for the out-of-plane modes as well as for the in-plane modes [24].

The topological invariant or Chern number defined for the integer quantum Hall system the TKNN [35, 36] invariant is closely related to the Berry phase. We have derived the TKNN invariant by calculating the Hall conductivity of a two-dimensional (2D) electron system in perpendicular magnetic field where the electric field \( E \) and the magnetic field \( B \) were applied along the \( y \) and \( z \) axes respectively.

The Heisenberg equation of motion \( \frac{\partial}{\partial y} v = \nu_y = \frac{i}{\hbar} [y, H] \) leads to [36]

\[
(m|v_y|n) = \frac{i}{\hbar}(E_n - E_m)(m|y|n),
\]

which allows one to calculate the Hall conductivity [36]

\[
\sigma_{xy} = \frac{\mu_e}{e^2} \sum_{n \neq m} f(E_n)(m|v_y|n)(m|v_y|n) - (m|v_y|n)(m|v_y|n),
\]

where \( v_y \) is the electron velocity along the \( x \) direction and \( f(E_n) \) is the Fermi distribution function. If we study a system in the periodic potential and its Bloch states \( |u_{nk}\rangle \) as the eigenstates the identity [36]

\[
\langle u_{nk}|v_y|u_{nk}\rangle = \frac{\hbar}{2}(E_n - E_m)|u_{nk}\rangle \langle u_{nk}|\frac{\hat{p}}{\hbar} |u_{nk}\rangle,
\]

allows one to rewrite [36]

\[
\sigma_{xy} = \frac{\mu_e}{e^2} \sum_{k} \sum_{n \neq m} f(E_n)(\frac{\partial}{\partial k_y} u_{nk} - \frac{\partial}{\partial k_y} u_{nk}) - \frac{\partial}{\partial k_y} u_{nk} - \frac{\partial}{\partial k_y} u_{nk},
\]

the Berry connection can be written as Bloch states [36]

\[
a_n(k) = -i(u_{nk}|\nabla k |u_{nk}) = -i(u_{nk}|\frac{\partial}{\partial k} |u_{nk}),
\]

the Hall conductivity reduces to \( \sigma_{xy} = \nu \frac{\hbar}{e} \), with

\[
\nu = \nu' = \sum_{n} \nu_n \text{ which was connected with the Berry phase [36]}
\]

\[
\nu_n = \int_{BZ} \frac{d^2 k}{2\pi}(\frac{\partial a_{x,y}}{\partial k_x} - \frac{\partial a_{x,y}}{\partial k_y}) = \frac{1}{2\pi} \int_{BZ} d^2 k a_n(k) = \frac{1}{2\pi} \gamma_n[\partial BZ],
\]

hence \( \gamma_n[\partial BZ] = 2\pi m, (m \in Z) [36] \).

Hence \( v_y \) can only take an integer value and hence \( \sigma_{xy} \) is quantized to integer multiples of \( \frac{\nu}{e^2} \). The integers \( \nu = \nu' \) were called TKNN invariant or Chern number and it plays the role of the topological invariant of the quantum spin Hall effect is represented by adding a mass term \( \Delta = \frac{\nu}{2} \sigma_z [22] \).

In the basis of the two-dimensional (2D) irreducible representation \( E \) for \( C_3v \) point symmetry the effective model takes the form of the 2D Weyl model for PtCl\(_3\) [22]

\[
\hat{H}_0(q) = v_F(\tau q_x \hat{\sigma}_x + q_y \hat{\sigma}_y),
\]

where \( v_F \) is the Fermi velocity, \( \tau = \pm \) for the \( K \) and \( K' \) point, \( \hat{\sigma} \) are the Pauli matrices acting in the space of the two basis states, and for the MoS\(_2\) [21]

\[
\hat{H}_0(q) = a(\tau q_x \hat{\sigma}_x + q_y \hat{\sigma}_y) + \frac{\lambda}{2} \hat{\sigma}_z - \lambda \tau \hat{\sigma}_z - \frac{1}{2} \hat{\sigma}_z,
\]

where \( \epsilon_z = \pm \frac{1}{2} \), \( \lambda \) is spin-valley splitting of valence band in the basis of the two-dimensional (2D) irreducible representation two valued \( \Gamma_7 \) for \( C_{3h} \) point symmetry. Hence we can conclude [22] that the ground state of monolayer PtCl\(_3\) indeed realizes a 2D WHS with a pair of fully polarized fermions at Weyl points robust under SOC.

A prominent property of Dirac fermions is that they carry the Berry phase of \( \pm \pi \). Hence the Chern number \( \nu' = \pm 1 \). Let us known 2D massless Weyl fermions with the Fermi velocity \( v_F \) for which the 4 \( \times \) 4 Dirac equation reduces to 2 \( \times \) 2 Weyl equation [36]

\[
E \psi(x) = \hbar v_F \hat{p} \psi(x) = -i\hbar v_F \hat{\sigma} \nabla \psi(x),
\]

the eigenvectors of this equation are

\[
\psi_\pm(x) = \begin{pmatrix} \exp^{-i\theta/2} \\ \pm \exp^{-i\theta/2} \end{pmatrix} = u_\pm(k) \exp^{ikx},
\]
where $\theta(k) = \arctan \frac{k_z}{k_x}$ and the energy eigenvalue are

$$E_\pm = \pm \hbar v_F k.$$  \hfill (12)

Keeping in mind the Stocks theorem one can find [36]

$$\gamma = \oint_{\gamma} d\mathbf{k} \mathbf{(u}_\pm(n(k)|\nabla u_\pm(k)) = \pm \pi.$$  \hfill (13)

We have found that Landau quantization occurs in the manner [36]

$$E_\pm(N) = \pm \sqrt(2\hbar v_F B/c)N,$$  \hfill (14)

where $N = 0, 1, 2, 3, \ldots$.

If the Landau quantization of the form Eq. (14) is true the associated QHE becomes unusual $\sigma_{xy} = -\frac{e^2}{h}(N + \frac{1}{2})$ which is called half-integer quantization. Hence the half-integer quantization can also be understood to be a result of the $\pm \pi$ Berry phases or $\pm 1$ Chern numbers [36].

The two QAH phases with $\mathcal{C} = \pm 1$ have edge channels propagating in opposite directions. Because the chirality of the edge channel is determined by the sign of Chern number.

Each atom has three first neighbors in the other sublattice i.e. $B$ with the relative vectors and six second neighbors in the same sublattice $A$ with the relative vectors too

$$B_1 = a(1, 0), \ B_2 = a(-\frac{1}{2}, \frac{\sqrt{3}}{2}), \ B_3 = a(-\frac{1}{2}, -\frac{\sqrt{3}}{2}),$$

$$A_1 = a(0, \sqrt{3}), \ A_2 = a(-\frac{3}{2}, \frac{\sqrt{3}}{2}), \ A_3 = -a(\frac{1}{2}, \frac{\sqrt{3}}{2}),$$

$$A_4 = -a(0, -\sqrt{3}), \ A_5 = -a(-\frac{3}{2}, \frac{\sqrt{3}}{2}), \ A_6 = a(\frac{1}{2}, \frac{\sqrt{3}}{2}).$$  \hfill (15)

The motion equations system in harmonic approximation we seek as follows [24]

$$\sum_{j/m,k'} \Phi^{\kappa\kappa'}_{ij}(a_n - a_m)u^{\kappa'}_{ij}(a_m) - \omega^2 u^\kappa_{ij}(a_n) = 0,$$

where the vectors $a_n$ numerate the lattice cells the $\kappa$ and $\kappa'$ note two sublattices $A$ and $B$ and the $i, j = x, y, z$ accounts three values corresponding space coordinates. Since the potential energy is the quadratic function of the atomic displacements $u^\kappa_i(a_n)$ and $u^\kappa_j(a_n)$, the forth constant matrix can be taken in the symmetrical form

$$\Phi^{\kappa\kappa'}_{ij}(a_n) = \Phi^{\kappa\kappa'}_{ji}(-a_n),$$  \hfill (17)

and its Fourier transform i.e. the dynamical matrix is a Hermitian matrix. For the nearest neighbors (in the $B$ sublattice) the dynamical matrix has the form

$$\Phi^{AB}_{ij}(q) = \sum_{k=1}^{3} \Phi^{AB}_{ij}(B_k) \exp(iqB_k),$$  \hfill (18)

and for the next neighbors (in the $A$ sublattice)

$$\Phi^{AA}_{ij}(q) = \Phi^{AA}_{ij}(A_0) + \sum_{k=1}^{6} \Phi^{AA}_{ij}(A_k) \exp(iqA_k),$$  \hfill (19)

where $A_0$ indices the atom chosen at the center of the coordinate system in the $A$ sublattice and $q$ is the wave vector.

The constants

$$\alpha_z \equiv \Phi^{AA}_{zz}(B_1),$$

$$\gamma_z \equiv \Phi^{AA}_{zz}(A_1).$$  \hfill (20)

We have found the stability condition

$$\Phi^{AA}_{zz}(A_0) + 6\Phi^{AA}_{zz}(A_1) + 3\Phi^{AA}_{zz}(B_1) = 0,$$  \hfill (22)

and the similar form for the $\xi \eta$ components, where off-diagonal elements are given by

$$\Phi^{AB}_{ij}(q) = \Phi^{AB}_{ij}(B_1) \exp(iQB_1) + \Phi^{AB}_{ij}(B_2) \exp(iQB_2) + \Phi^{AB}_{ij}(B_3) \exp(iQB_3) =$$

$$= \Phi^{AB}_{ij}(B_1) \exp(iq_1 + \exp^{-2i\pi/3} \exp^{-i2\pi/3} \exp^{-i2\pi/3} \exp^{i2\pi/3} \exp^{i2\pi/3} \exp^{-i2\pi/3} q_2) =$$

$$= \beta(\exp^{i\theta_1} + 2 \exp^{i\theta_1} \cos(\frac{\sqrt{3}}{2}q_y - \frac{2}{3}\pi)).$$  \hfill (23)
Figure 1. (Color online) Frequency dispersions or Phonon dispersion spectrums in MoS$_2$ from point $\Gamma$ (0,0) to M $(1, \sqrt{3})_{3d}$ and from point M $(1, \sqrt{3})_{3d}$ to K (0,1) $\frac{4\pi}{3\sqrt{3}a}$.

Figure 2. (Color online) Frequency dispersions or Phonon dispersion spectrums in MoS$_2$ from point K (0,1) $\frac{4\pi}{3\sqrt{3}a}$ to $\Gamma$ (0,0).

Figure 3. (Color online) Frequency dispersions or Phonon dispersion spectrums in MoS$_2$ from point $\Gamma$ (0,0) to M $(1, \sqrt{3})_{3d}$ and from point M $(1, \sqrt{3})_{3d}$ to K (0,1) $\frac{4\pi}{3\sqrt{3}a}$. 
Figure 4. (Color online) Frequency dispersions or Phonon dispersion spectrums in MoS$_2$ from point K (0,1) $\frac{4\pi}{3\sqrt{3}a}$ to $\Gamma$ (0,0).

Figure 5. (Color online) Frequency dispersions or Phonon dispersion spectrums in PCl$_3$ from point $\Gamma$ (0,0) to $M$ $(1, \sqrt{3})\frac{a}{3a}$ and from point $M$ $(1, \sqrt{3})\frac{a}{3a}$ to K (0,1) $\frac{4\pi}{3\sqrt{3}a}$.

Figure 6. (Color online) Frequency dispersions or Phonon dispersion spectrums in PCl$_3$ from point K (0,1) $\frac{4\pi}{3\sqrt{3}a}$ to $\Gamma$ (0,0).
Figure 7. (Color online) Frequency dispersions or Phonon dispersion spectrums in PlCl₃ from point Γ (0,0) to M (1, √3/3) and from point M (1, √3/3) to K (0,1) $\frac{4\pi}{3\sqrt{3}c}$. 

Figure 8. (Color online) Frequency dispersions or Phonon dispersion spectrums in PlCl₃ from point K (0,1) $\frac{4\pi}{3\sqrt{3}c}$ to Γ (0,0).
Figure 9. (Color online) Figures from Phys. Rev. B 100 (064408) 2019.
\[\Phi^{AB}(A_0) = \Phi^{AA}(A_0) \exp^{i(\mathbf{q}A_0)} + \Phi^{BB}(A_0) \exp^{i(\mathbf{q}A_0)} \]

The point group \(D_{6h}\) of the honeycomb lattice is generated by \(\{C_6, \sigma_x, \sigma_z\}\) where \(\sigma_x\) is a reflection \(z \rightarrow -z\) by the plane that contains the graphene layer, \(C_6\) is a rotation by \(\pi/3\) around the \(z\) axis and \(\sigma_z\) is a reflection by the \(xz\) plane. To find the dynamic matrix we introduce variables \(\xi, \eta = x \pm iy\) transforming under the rotation \(C_6\) around the \(z\)-axis (taken at the \(A_0\) atom) as follows \((\xi, \eta) \rightarrow (\xi, \eta) \exp^{i(\pm2\pi/3)}\). At the rotation the atoms change their positions \(B_1 \rightarrow B_2 \rightarrow B_3, A_1 \rightarrow A_3 \rightarrow A_5\) and \(A_2 \rightarrow A_4 \rightarrow A_6\). Hence all forth constant \(\Phi^{AB}(B_6)\) with the different \(\kappa\) (as well as \(\Phi^{AB}(B_6)\)) are equal to one another but the force constants with the coincident subscripts \(\xi\) or \(\eta\) transform as covariant variables

\[\begin{align*}
\Phi^{AB}(B_1) = & \Phi^{AB}(B_2) \exp^{2i\pi/3} = \Phi^{AB}(B_3) \exp^{-2i\pi/3}, \\
\Phi^{AA}(A_1) = & \Phi^{AB}(A_2) \exp^{-2i\pi/3} = \Phi^{AB}(A_3) \exp^{2i\pi/3}, \\
\Phi^{BB}(A_4) = & \Phi^{AB}(A_6) \exp^{i\pi/3}.
\end{align*}\]

We have taken in account the this stability condition also

\[\Phi^{AA}(A_0) + 6\Phi^{AA}(A_1) + 3\Phi^{AB}(B_1) = 0.\]

The dynamical matrix for the in-plane oscillations for the honeycomb lattice with two atoms per unit cell for \(N \times N\) unit cells has the form

\[\begin{pmatrix}
\Phi^{AA}(q) & \Phi^{AB}(q) \\
\Phi^{AB} & \Phi^{BB}(q)
\end{pmatrix}
\]

where

\[\beta(\exp^{i\eta} + 2 \exp^{i\pi/3} \cos(\frac{\sqrt{3}}{2}q_y - \frac{\pi}{3}) = \beta(\cos(q_x) + i \sin(q_x) + 2(\cos(\frac{\pi}{2}) - i \sin(\frac{\pi}{2}))\cos(\frac{\sqrt{3}}{2}q_y - \frac{\pi}{3}) = \]

\[= \beta(\cos(q_x) + 2 \cos(\frac{\pi}{2}) \cos(\frac{\sqrt{3}}{2}q_y - \frac{\pi}{3}) + i \sin(q_x) - 2 \sin(\frac{\pi}{2}) \cos(\frac{\sqrt{3}}{2}q_y - \frac{\pi}{3})).\]
\[ \delta(\exp^{\sqrt{3}q_y} + 2\exp^{-\frac{i}{2}q_y} \cos(\frac{1}{2}q_x + \frac{2}{3}\pi)) + \delta'(\exp^{-\sqrt{3}q_y} + 2\exp^{\frac{i}{2}q_y} \cos(\frac{1}{2}q_x - \frac{2}{3}\pi)) = \]
\[ = \delta(\cos(\sqrt{3}q_y) + i\sin(\sqrt{3}q_y) + 2(\cos(\frac{\sqrt{3}}{2}q_y) - i\sin(\frac{\sqrt{3}}{2}q_y))) + \]
\[ + \delta'(\cos(\sqrt{3}q_y) - i\sin(\sqrt{3}q_y) + 2(\cos(\frac{\sqrt{3}}{2}q_y) + i\sin(\frac{\sqrt{3}}{2}q_y))) = \]
\[ = \delta(\cos(\sqrt{3}q_y) + 2(\cos(\frac{\sqrt{3}}{2}q_y) \cos(\frac{1}{2}q_x + \frac{2}{3}\pi)) + \]
\[ + \delta'(\cos(\sqrt{3}q_y) + 2(\cos(\frac{\sqrt{3}}{2}q_y) \cos(\frac{1}{2}q_x - \frac{2}{3}\pi)) = \]
\[ = \delta(\cos(\sqrt{3}q_y) + 2(\cos(\frac{\sqrt{3}}{2}q_y) \cos(\frac{1}{2}q_x + \frac{2}{3}\pi)) + \]
\[ + \delta'(\cos(\sqrt{3}q_y) + 2(\cos(\frac{\sqrt{3}}{2}q_y) \cos(\frac{1}{2}q_x - \frac{2}{3}\pi)) = \]
\[ = \delta(\cos(\sqrt{3}q_y) + 2(\cos(\frac{\sqrt{3}}{2}q_y) \cos(\frac{1}{2}q_x + \frac{2}{3}\pi)) + \]
\[ + \delta'(\cos(\sqrt{3}q_y) + 2(\cos(\frac{\sqrt{3}}{2}q_y) \cos(\frac{1}{2}q_x - \frac{2}{3}\pi)) = 0 \]}

We have found the matrix factor for the equation of motion in the form

\[ \Phi^{\alpha}(q) = \begin{pmatrix}
\Phi^{\alpha}_{x}(q) - \omega^2 \\
\Phi^{\alpha}_{y}(q) - \omega^2 \\
\Phi^{\alpha}_{z}(q) - \omega^2
\end{pmatrix}, \]

where

\[ \Phi^{\alpha}_{x}(q) = \beta(\exp^{i\alpha} + 2\exp^{-i\alpha} \cos(1/2q_y - \frac{2}{3}\pi)), \]

\[ \Phi^{\alpha}_{y}(q) = \alpha_x(\exp^{i\alpha} - \exp^{-i\alpha} \cos(1/2q_y)), \]

\[ \Phi^{\alpha}_{z}(q) = \Phi^{\alpha}_{x}(B_1)(\exp(i\alpha) + \exp(i\alpha)) - \alpha_x(\exp^{i\alpha} - \exp^{-i\alpha} \cos(1/2q_y)), \]

\[ \Phi^{\alpha}(q) = \Phi^{\alpha}_{x}(A_0) + \Phi^{\alpha}_{y}(A_1)(\exp(i\alpha) + \exp(i\alpha)) + \exp(i\alpha) + \exp(i\alpha) + \exp(i\alpha) + \exp(i\alpha) = \]
\[ = \alpha_x(\exp^{i\alpha} + \exp^{-i\alpha} \cos(1/2q_y)), \]

\[ = -6\gamma_x - 3\alpha_x + \gamma_x(2\cos(\sqrt{3}q_y) + 2\exp(i\alpha) \cos(1/2q_y)) = \]
\[ = -6\gamma_x - 3\alpha_x + \gamma_x(2\cos(\sqrt{3}q_y) + 2\exp(i\alpha) \cos(1/2q_y)) = \]
\[ = -3\alpha_x + \gamma_x(3\cos(\sqrt{3}q_y) + 2\cos(1/2q_y) \cos(1/2q_y) - 3), \]

\[ \alpha_x(\exp^{i\alpha} + \exp^{-i\alpha} \cos(1/2q_y)) = \alpha_x(1 + \frac{i}{2}q_y + 2(1 - \frac{i}{2}q_y)(1 + (\frac{\sqrt{3}}{2}q_y)^2)) = \]
\[ = \alpha_x(1 + \frac{i}{2}q_y + 2(1 - \frac{i}{2}q_y)(1 + (\frac{\sqrt{3}}{2}q_y)^2)) = \alpha_x(1 + \frac{i}{2}q_y + 2(1 - \frac{i}{2}q_y)(1 + (\frac{\sqrt{3}}{2}q_y)^2)) = \]
\[ = \alpha_x(1 + \frac{i}{2}q_y + 2(1 - \frac{i}{2}q_y)(1 + (\frac{\sqrt{3}}{2}q_y)^2)) = \alpha_x(3 + \frac{3}{2}q_y^2) = 3\alpha_x(1 + \frac{q_y^2}{2}). \]

The phonon dispersion for the out-of-plane modes is found the equations of motion

\[ \det \begin{vmatrix}
\Phi^{\alpha}_{x}(q) - \omega^2 \\
\Phi^{\alpha}_{y}(q) - \omega^2 \\
\Phi^{\alpha}_{z}(q) - \omega^2
\end{vmatrix} = 0, \]

\[ (\Phi^{\alpha}_{x}(q) - \omega^2)^2 - |\Phi^{\alpha}_{x}(q)|^2 = 0, \]

\[ \Phi^{\alpha}_{x}(q) - \omega^2 = \pm|\Phi^{\alpha}_{x}(q)|, \]

\[ \Phi^{\alpha}_{x}(q) = |\Phi^{\alpha}_{x}(q)| = \omega^2, \]

\[ \omega(q) = \sqrt{\Phi^{\alpha}_{x}(q) + |\Phi^{\alpha}_{x}(q)|}. \]
\[ \omega(q_y = q, q_x = \frac{\alpha}{\sqrt{3}}) = \sqrt{\Phi_{zz}^{\text{AM}}(q_y = q, q_x = \frac{\alpha}{\sqrt{3}})} \pm |\Phi_{zz}^{\text{AB}}(q_y = q, q_x = \frac{\alpha}{\sqrt{3}})|, \]

\[ k_y = 0..\frac{\sqrt{3}\pi}{3a}, \]

\[ M - K, \]

\[ q_y = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}, \]

\[ q_y = q, \]

\[ q_x = \sqrt{3}q, \]

\[ \omega(q_y = q, q_x = \sqrt{3}q) = \sqrt{\Phi_{zz}^{\text{AM}}(q_y = q, q_x = \sqrt{3}q)} \pm |\Phi_{zz}^{\text{AB}}(q_y = q, q_x = \sqrt{3}q)|, \]

\[ k_y = \frac{\sqrt{3}\pi}{3a}..\frac{4\pi}{3\sqrt{3}a}, \]

\[ K - \Gamma, \]

\[ q_y = q, \]

\[ q_x = 0, \]

\[ \omega(q_y = q, q_x = 0) = \sqrt{\Phi_{zz}^{\text{AM}}(q_y = q, q_x = 0)} \pm |\Phi_{zz}^{\text{AB}}(q_y = q, q_x = 0)|, \]

\[ k_y = \frac{4\pi}{3\sqrt{3}a}.0, \]

---

**Table 3.** The Force parameters of phonon oscillating matrices of graphene [24].

<table>
<thead>
<tr>
<th>( \alpha_x )</th>
<th>( \gamma_x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.27</td>
<td>0.204</td>
</tr>
</tbody>
</table>

\[ \alpha_x (\exp^{q_x} + 2\exp^{-\frac{iq_x}{2}} \cos \frac{\sqrt{3}q_y}{2}) = \alpha_x (\cos q_x + i \sin q_x + 2(\cos \frac{q_y}{2} - i \sin \frac{q_y}{2}) \cos \frac{\sqrt{3}q_y}{2}) = \alpha_x (\cos q_x + 2 \cos \frac{q_y}{2} \cos \frac{\sqrt{3}q_y}{2} + i(\sin q_x - 2 \sin \frac{q_y}{2} \cos \frac{\sqrt{3}q_y}{2})), \]

\[ |\alpha_x (\cos q_x + 2 \cos \frac{q_y}{2} \cos \frac{\sqrt{3}q_y}{2} + i(\sin q_x - 2 \sin \frac{q_y}{2} \cos \frac{\sqrt{3}q_y}{2}))) = \alpha_x \sqrt{(\cos q_x + 2 \cos \frac{q_y}{2} \cos \frac{\sqrt{3}q_y}{2} + i(\sin q_x - 2 \sin \frac{q_y}{2} \cos \frac{\sqrt{3}q_y}{2})^2 + (\sin q_x - 2 \sin \frac{q_y}{2} \cos \frac{\sqrt{3}q_y}{2})^2}, \]

So we found the three acoustic branches and optical branches which are in close agreement with Ref. [24, 22, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47]. In the framework our calculations one can conclude that in the K point the acoustic and optical branches have Dirac point similarly to the Dirac cone in graphene and light cone in relativistic physics, where the light velocity is substituted by the phonon velocity and describes by the Dirac equation. In Fig. 9 we have presented (a) schematic depiction of the orbital splitting in monolayer PtCl3. In (b) Spin-resolved partial density of states (PDOS) for monolayer PtCl3 projected on different orbitals. In (c) band structure without spin-orbit coupling (SOC). The red and blue bands are for spin-majority (spin-up) and spin-minority (spin-down) channels respectively. In (d) enlarged view of the band structure around the Weyl point. The red solid (blue dashed) lines are...
located at K/K' points without SOC (blue points), and they are shifted along the x direction on the mirror-invariant line after considering SOC (red points) [22]. In (a) top and side views of monolayer PtCl₄, with the edge-sharing PtCl₃ octahedron forming a honeycomb lattice. In (b) first Brillouin zone for monolayer PtCl₄ with high-symmetry $D_{3d}$ point group symmetry. We also mark the three vertical mirror planes for the lattice structure (red lines) [22].

Let us known the phonon dispersions and their wave vectors from the equations of motion

$$
\Phi_{\alpha\xi}(q) - \omega^2 = 0,
\Phi_{\alpha\eta}(q) - \omega^2 = 0,
\Phi_{\alpha\xi}(q) + \Phi_{\alpha\eta}(q) u_{\xi}(q) = 0,
\Phi_{\alpha\eta}(q) + \Phi_{\alpha\xi}(q) u_{\eta}(q) = 0,
$$

(60)

$$
\omega_+ : u_{\alpha}(q) = -\frac{\Phi_{\alpha\xi}(q)}{\Phi_{\alpha\eta}(q) - \omega^2} u_{\eta}(q),
$$

(61)

$$
\Phi_{\alpha\xi}(q) u_{\alpha}(q) + (\Phi_{\alpha\eta}(q) - \omega^2) u_{\eta}(q) = 0,
$$

(62)

with orthonormalized condition of wave vectors of out-of plane modes of coherent phonon oscillation of lattice

$$
|u_{\alpha}(q)|^2 + |u_{\eta}(q)|^2 = 1,
$$

(63)

$$
|\Phi_{\alpha\xi}(q)|^2 |u_{\eta}(q)|^2 + |u_{\eta}(q)|^2 = 1,
$$

(64)

$$
u_{\eta}(q) = \frac{1}{\sqrt{1 + |\Phi_{\alpha\xi}(q)|^2}} = \frac{1}{\delta},
$$

(65)

$$
\Phi_{\alpha\xi}(q) u_{\alpha}(q) + (\Phi_{\alpha\xi}(q) - \omega^2) u_{\eta}(q) = 0,
$$

(66)

$$
u_{\xi}(q) = -\frac{\Phi_{\alpha\eta}(q)}{\Phi_{\alpha\xi}(q) - \omega^2} u_{\xi}(q),
$$

(67)

$$
|u_{\alpha}(q)|^2 + |u_{\xi}(q)|^2 = 1,
$$

(68)

$$
|u_{\xi}(q)|^2 + |u_{\eta}(q)|^2 = 1,
$$

(69)

$$
|u_{\alpha}(q)|^2 + |u_{\xi}(q)|^2 = 1,
$$

(70)

$$
u_{\alpha}(q) = \frac{1}{\sqrt{1 + |\Phi_{\alpha\xi}(q)|^2}} = \frac{1}{\delta},
$$

(71)

The phonon dispersions and their wave vectors for in-plane coherent phonon branches of oscillation of lattice

$$
\Phi_{\alpha\xi}(q) - \omega^2 = 0,
\Phi_{\alpha\eta}(q) - \omega^2 = 0,
\Phi_{\alpha\xi}(q) + \Phi_{\alpha\eta}(q) u_{\xi}(q) = 0,
\Phi_{\alpha\eta}(q) + \Phi_{\alpha\xi}(q) u_{\eta}(q) = 0,
$$

(73)

Hence

$$
\Phi_{\alpha\xi}(q) u_{\alpha}(q) + \Phi_{\alpha\eta}(q) u_{\eta}(q) = 0,
\Phi_{\alpha\eta}(q) u_{\alpha}(q) + \Phi_{\alpha\xi}(q) u_{\xi}(q) = 0,
\Phi_{\alpha\xi}(q) u_{\eta}(q) + \Phi_{\alpha\eta}(q) u_{\xi}(q) = 0,
\Phi_{\alpha\eta}(q) u_{\eta}(q) + \Phi_{\alpha\xi}(q) u_{\xi}(q) = 0,
$$

(74)
\[ \begin{align*}
\Phi_{\alpha}(q) &= \frac{\Phi_{\alpha}(q)}{\Phi_{\alpha}(q)_{\text{c}}} \left( -\Phi_{\alpha}(q)_{\text{c}} \right) + \Phi_{\alpha}(q)_{\text{c}}, \\
\Phi_{\beta}(q) &= \frac{\Phi_{\beta}(q)}{\Phi_{\beta}(q)_{\text{c}}} \left( -\Phi_{\beta}(q)_{\text{c}} \right) + \Phi_{\beta}(q)_{\text{c}}, \\
\Phi_{\gamma}(q) &= \frac{\Phi_{\gamma}(q)}{\Phi_{\gamma}(q)_{\text{c}}} \left( -\Phi_{\gamma}(q)_{\text{c}} \right) + \Phi_{\gamma}(q)_{\text{c}}, \\
\Phi_{\delta}(q) &= \frac{\Phi_{\delta}(q)}{\Phi_{\delta}(q)_{\text{c}}} \left( -\Phi_{\delta}(q)_{\text{c}} \right) + \Phi_{\delta}(q)_{\text{c}}, \\
\Phi_{\epsilon}(q) &= \frac{\Phi_{\epsilon}(q)}{\Phi_{\epsilon}(q)_{\text{c}}} \left( -\Phi_{\epsilon}(q)_{\text{c}} \right) + \Phi_{\epsilon}(q)_{\text{c}}. 
\end{align*} \]
Eqs. (103), (102), (101) from (100) like with orthonormalized condition of wave vectors for in-plane modes of coherent phonon branches of oscillations of lattice allows the creation of the 2D WHS states with the pairs of fully polarized fermions at Weyl points which robust under

We seek linearly independent solutions of wave vectors of coherent phonon oscillations

\[
\begin{align*}
\psi_1(q) &= \frac{1}{\alpha} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \\
\psi_2(q) &= \frac{1}{\alpha} \begin{bmatrix} 1 \\ 1/\beta \\ 1 \\ 0 \end{bmatrix}, \\
\psi_3(q) &= \frac{1}{\alpha} \begin{bmatrix} 1 \\ 1/\beta \\ 1/\delta \\ 1 \end{bmatrix}, \\
\psi_4(q) &= \frac{1}{\alpha} \begin{bmatrix} 1 \\ 1/\beta \\ 1/\delta \\ 1 \end{bmatrix},
\end{align*}
\]

We seek linearly independent solutions of wave vectors of coherent phonon oscillations [48, 49, 50, 41, 51, 52, 53] Eqs. (103), (102), (101) from (100) like A → B, B → A, * → o, o → *.

Hence we have found that the Weyl Hall semiconductor (WHS) states allow the topological phase transition happened between two quantum anomalous Hall (QAH) insulator phase with opposite Chern numbers and we have based on phonon dispersion of Weil 2D half semiconductor monolayers group like PtCl$_3$. The ground state of monolayer PtCl$_3$ allows the creation of the 2D WHS states with the pairs of fully polarized fermions at Weyl points which robust under
spin-orbit coupling (SOC). We have proposed the switching of the quantum anomalous Hall states by rotating the magnetization vector on the basis of PtCl₃ monolayer. So we have proposed the switchable QAH phases will enable new designs of topological nanoelectronic devices in basis on group materials like 2D PtCl₃ type [54, 55, 56, 26, 57, 58, 59, 60, 61, 62, 63, 64, 65].

4 Optical properties of crystals with Incommensurate phases from first principle

Refractive indexes can be determined like the eigenvalues of the dielectric permeability tensor as follows [4]

\[ \det |\epsilon_{ab}(\omega, k = 0) - n(\omega)\delta_{ab}| = 0. \] (104)

The tensor \( \epsilon_{ab}(\omega, k = 0) \) is expressed by the two-time retarded Green functions [4],

\[ \epsilon_{ab}(\omega, k = 0) = \delta_{ab} - \frac{8\pi^2}{\hbar^2} \sum_{mn} \langle \hat{p}_{n}^a \hat{p}_{n'}^b \rangle, \] (105)

where \( \hat{p}_{n}^a \) are the operators of the electric dipole momenta of the unit cells of a crystal.

Let us know the case of ionic type insulator materials. Their optical properties in the short wave part of the diffraction transparent region were determined by the electronic dipole transitions with the separate ions and ionic groups with accounts coherent phonon oscillations. In the dipole momenta the electronic and ionic components which these that account for phonon oscillations of crystal lattices were separated [4, 9, 66, 67, 68, 69],

\[ \hat{p}_{n}^a = \sum_{k}(e \hat{D}_{nk}^a + Z_e u_{nk}^a), \] (106)

where \( e \) and \( Z_e \) are charges of electron and ion, respectively,

\[ \hat{D}_{nk}^a = \sum_{\alpha} \mu_{\kappa \alpha} \hat{u}_{nk}^\alpha, \] (107)

is the operator of electron coordinates of the \( \kappa \)-type ion, written in the representation of the Hubbard operators \( \hat{X}_{nk}^{\alpha'} = |nks\rangle \langle nks'| \) acting in the space of electron states \{\( \psi_{nks} \)\} of the ion, \( \mu_{\alpha'}^\kappa \) are the corresponding matrix elements, \( u_{nk}^\alpha \) the components of the ionic displacements from the equilibrium positions account of coherent phonon oscillations of crystal lattices. If we take into consideration the electron excitations only then

\[ \langle \langle \hat{p}_{n}^a \hat{p}_{n'}^b \rangle \rangle_\omega = e^2 \sum_{k, k', \alpha, \beta} \langle \langle \hat{D}_{nk}^a \hat{D}_{nk'}^b \rangle \rangle_\omega. \] (108)

In the article [4] for ionic insulator crystals the Hamiltonian of ionic crystal with account electronic excitation of ionic or ionic complexes as well as phonon lattice vibration oscillations at expansion of interactions in the sets with respect to ionic displacements as well as electric dipole momenta has been presented via Hubbard operators \( \hat{X}_{nk}^{\alpha'} = |nks\rangle \langle nks'| \) from which consist of the basis electronic states of \( \psi_{nks} \) ion in the form [4, 19, 70, 71, 72, 73, 74, 75, 76, 77, 78]

\[ H = \sum_{nks} \lambda_{ks} \hat{X}_{nk}^{\alpha'} - \sum_{nka,n'k} \langle \langle \psi_{nka}^{\alpha'}(nk,n'k)\hat{p}_{nk}^{a}\hat{p}_{nk'}^{b} + C_{nk}(nk,n'k)\hat{D}_{nk}^{a}u_{nk}^{\alpha'}+ \sum_{mn'k'} \theta_{\alpha\beta\gamma}(nk,n'k',n''k'')\hat{D}_{nk}^{a}u_{nk'}^{\alpha'}u_{nk''}^{\beta}u_{nk}^{\gamma} + \sum_{mn'k'k''} \theta_{\alpha\beta\gamma}(nk,n'k',n''k'',n'''k''')\hat{D}_{nk}^{a}u_{nk'}^{\alpha'}u_{nk''}^{\beta}u_{nk'''}^{\gamma}u_{nk}^{\gamma} \rangle \rangle_{\omega}. \] (109)

where \( u_{nk}^\alpha \) are ion displacement vectors accordingly equilibrium states, \( \hat{D}_{nk}^a \) is ion dipole momenta, \( \lambda_{ks} \) is energy of s state of ion of k nature.

For the equilibrium positions \( \mathbf{R}^{(0)} = \mathbf{n} + \mathbf{r}_k \) electron wave functions \( \psi_{nks} \) and energies \( \lambda_{ks} \) the ones in the high-temperature commensurate phase of the crystal are chosen.

The structure of the incommensurate phase is given by the average displacements and average dipole momenta which may be presented in the form of modulation waves [4]

\[ u_{nk}^\alpha = \langle u_{nk}^\alpha \rangle + \tilde{u}_{nk}^\alpha, \] \[ D_{nk}^a = \langle D_{nk}^a \rangle + \tilde{D}_{nk}^a, \] (110)

where \( \langle u \rangle \) and \( \langle D \rangle \) describe average displacements and average dipole momenta ions (\( \langle u \rangle = 0 \) and \( \langle D \rangle = 0 \)) in high temperature commensurate phases as well as \( \langle u \rangle \neq 0 \) and \( \langle D \rangle \neq 0 \) in incommensurate phases in low temperature of phase transition and \( \tilde{u}_{nk}^\alpha \) as well as \( \tilde{D}_{nk}^a \) were deviations from average values.

In order to deposit the electronic subsystem Hamiltonian [4] we have entered main field approximation [25, 79, 80]

\[ \hat{H}_{MF}^{el} = \sum_{nk} \sum_{\alpha} \langle \lambda_{ks} \hat{D}_{nk}^a - \sum_{a} F_{nk}^a \mu_{\alpha} \hat{X}_{nk}^{\alpha'} \rangle \hat{X}_{nk}^{\alpha'}, \] (111)

where
\[ F_n^a = \sum_{n'k'\beta}(\psi_{a\beta}(nk,n'k')(D_{n'k'}^\beta) + C_{a\beta}(nk,n'k')(u_{n'k'}^\beta) - \sum_{n'k',n''k''\gamma}(\sigma_{a\beta\gamma}(nk,n'k',n''k'')(D_{n'k''}^\beta)(u_{n''k''}^\beta) + (D_{n'k'k''}^\beta)(u_{n'k''}^\beta) + \sigma_{a\beta\gamma}(nk,n'k',n''k'')(u_{n'k'}^\beta)(u_{n''k''}^\beta) + (\tilde{u}_{n'k'}^\beta(\tilde{u}_{n''k''}^\beta)))). \]

It is known that the Hamiltonian \( \hat{H}_{MF}^a \) describes the influence crystal field of lattice \( F_n^a \) on electronic states of ions [16, 81, 82, 83, 84]. In order to obtain from the Hamiltonian \( \hat{H}_{MF}^a \) the diagonal matrices we have considered the unitary transformation

\[ \hat{X}_{nk} = \sum_{\mu\nu} u_{nk}^{\mu\nu} \hat{X}_{nk}^{\mu\nu}, \]

where \( u_{nk}^{\mu\nu} \) were eigenvectors which were found from equations system

\[ \sum_{\mu}(\lambda_{nk} \delta_{\mu\nu} - \sum_{\nu} F_{nk}^a u_{nk}^{\nu\mu}) u_{nk}^{\mu\nu} = \tilde{\lambda}_{nk} u_{nk}^{\mu\nu}. \]

Hence

\[ \hat{H}_{MF}^a = \sum_{\mu\nu} \tilde{\lambda}_{nk} \hat{X}_{nk}^{\mu\nu}, \]

where \( \tilde{\lambda}_{nk} \) were new energy levels splitting by Stark effect [85, 86, 82, 87, 88, 89, 90, 91]. We have transformed \( \hat{D}_{nk}^a \) operators into the form

\[ \hat{D}_{nk}^a = \sum_{\mu\nu} \tilde{\mu}_{nk}^{\mu\nu} \hat{X}_{nk}^{\mu\nu}, \]

\[ \tilde{\mu}_{nk}^{\mu\nu} = \sum_{\mu\nu} \hat{X}_{nk}^{\mu\nu} \tilde{X}_{nk}^{\mu\nu}. \]

We have presented dipole-dipole Green function on the based Eqs. (116), (117) in the form

\[ \langle \langle \hat{D}_{nk}^a | \hat{D}_{nk'}^a \rangle \rangle = \sum_{\mu\nu\mu'\nu'} F_{nk}^a \delta_{\mu\nu} \delta_{\mu'\nu'} \langle \langle \hat{X}_{nk}^{\mu\nu}, \hat{X}_{nk'}^{\mu'\nu'} \rangle \rangle. \]

The Green function \( \langle \langle \hat{X}\hat{X} \rangle \rangle \) satisfy the motion equation

\[ (\hbar \omega \tilde{\lambda}_{nk} + \tilde{\lambda}_{nk}) \langle \langle \hat{X}_{nk}^{\mu\nu}, \hat{X}_{nk'}^{\mu'\nu'} \rangle \rangle = \frac{1}{2} \sum_{\gamma s} \delta_{\mu\nu} \delta_{\mu'\nu'} \langle \langle \hat{X}_{nk}^{\mu\nu}, \hat{X}_{nk'}^{\mu'\nu'} \rangle \rangle - \sum_{\gamma s} \tilde{\psi}_{\gamma s}^a(nk,n'k') \langle \langle \hat{D}_{nk}^a | \hat{X}_{nk'}^{\mu'\nu'} \rangle \rangle, \]

but

\[ \tilde{\psi}_{\gamma s}^a(nk,n'k') = \psi_{\gamma s}^a(nk,n'k'), \sum_{\gamma s} \tilde{\psi}_{\gamma s}^a(nk,n'k') - \sum_{\gamma s} \tilde{\psi}_{\gamma s}^a(nk,n'k'), \sum_{\gamma s} \tilde{\psi}_{\gamma s}^a(nk,n'k')(u_{nk}^{\mu\nu} - X_{nk}^{\mu\nu}) - \frac{1}{2} \sum_{\gamma s} \tilde{\psi}_{\gamma s}^a(nk,n'k') \sum_{\gamma s} \tilde{\psi}_{\gamma s}^a(nk,n'k'')(u_{nk}^{\mu\nu} - X_{nk}^{\mu\nu}), \]

Hence we have derived Tyablikov uncoupling \( \langle \langle \hat{D}\hat{u} \rangle \rangle \) as well as we have cut the expressions with \( \langle \langle \hat{D}\hat{u} \hat{X} \rangle \rangle \) Green function as well as we have uncoupled \( \langle \langle \hat{u}\hat{X} \rangle \rangle \) and \( \langle \langle \hat{D}\hat{D} \rangle \rangle \) in additions. Hence we have derived the Bethe-Salpeter equations [92, 93, 94] for Green function in the form

\[ \langle \langle \hat{D}_{nk}^a | \hat{D}_{nk'}^a \rangle \rangle = \frac{1}{2} \sum_{\gamma s} \tilde{\psi}_{\gamma s}^a(nk,n'k') \sum_{\gamma s} \tilde{\psi}_{\gamma s}^a(nk,n'k')(u_{nk}^{\mu\nu} - X_{nk}^{\mu\nu}), \]

where

\[ \tilde{Z}_{nk}^a(\omega) = \sum_{\mu\nu} \bar{u}_{nk}^{\mu\nu} \chi_{\nu\nu}^a(\omega) \langle \langle \hat{X}_{nk}^{\mu\nu}, \hat{X}_{nk}^{\mu\nu} \rangle \rangle, \]

accounts of electronic polarizability single ion displaced in the field \( F_n^a \) crystal lattice.

### 4.1 Symmetry of Incommensurate phase crystals

It is known from neutron and X-ray scattering that the phase transition from high temperature commensurate phase in incommensurate phase caused by equilibrium breaking in general defined \( q' \) point of Brillouin zone [4, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 80]. At these the created structure which displaced low temperature \( T_1 \) of phase transition were described for the phase transition of displacement type in K$_3$SeO$_4$ the modulation wave of ions displacement \( u_{nk} = u_l(q'n + \varphi) \) but for the phase transition order-orderless in NaNO$_2$ by the modulation wave of occupation probability of single ion the one from several equilibrium location \( p_{nk} = p_k(q'n + \varphi) \), where \( n = n_1, n_2, n_3 + n_3 \) was lattice vector of basic high temperature commensurate phase. The vectors \( u_l(\varphi) \) and \( p_k(\varphi) \) are periodic functions of phase \( \varphi \) with 2\( \pi \) period.
Hence like for the phase transition of displacement type thus for the phase transition of order-orderless the average displacement of ions [108, 25, 14, 109, 110, 111, 112] in incommensurate phase can be presented in the form

\[
\langle u_{nk} \rangle = C_k(q' n + \varphi),
\]

(123)

where \( C_k(\varphi) = C_k(\varphi + 2\pi) \).

Since the wave vector \( q' \) is incommensurate with inverse lattice vector, (i.e. does not exist of like these integer \( M \) and \( N \)) for which

\[
Nq' + Ma^* = 0,
\]

(124)

where \( a^* \) is inverse lattice vector.

For study Incommensurate phase symmetry the \( 3 + d \)-dimensional supersymmetry groups have been derived. For experimental observations sufficiently suggest \( d = 1 \) one-dimensional modulation. The supersymmetry group approach was build based on theoretical group analysis or tight-binding method. In which average displacements Eq. (123) were connected with order parameter of phase transition from commensurate phase into incommensurate phase

\[
\langle u_{nk} \rangle = \sum_{q,j} Q(q, j) c_q(q, j) \exp(iq(n+\tau_j)),
\]

(125)

where \( Q(q, j) \) order parameter which transforms like \( j \) irreducible representation for wave vector \( q, c_q(q, j) \) eigenvector this representation; \( \sum_{q,j} \) includes the contributions from main order parameter \( Q(q', j_0) \) corresponding representation \( j_0 \) for the wave vector \( q' \) as well as the contributions from secondary order parameters \( Q(q_i, j_i) \) for which the symmetry allows the existing in Landau expansion for free energy such terms which are proportional to

\[
Q^n(q', j_0)Q(-q_i, j_i)\Delta(n_iq' - q_i),
\]

(126)

where \( \Delta(k) = \sum_\alpha \delta(k - G) \), but \( G \) is inverse lattice vector.

Since

\[
Q(q', j_0) = Q^*(-q', j_0) = \eta \exp^{r'},
\]

(127)

and take into account the following connections \( Q(q_i, j_i) \sim Q^n(-q', j_0), q_i = n_iq' + m_ia^* \) which effluent from Eq. (126) can assure that found expansion for \( \langle u_{nk} \rangle \) satisfy the expression Eq. (123).

It is known that the symmetry of average ionic displacements satisfy the condition of invariance of four-dimensional lattice \( (n + r_k + C_q(q'n + \varphi), \varphi) \) with respect supersymmetry group operations. The \( \hat{g} \) elements of supersymmetry group have a form

\[
\hat{g} = ([\hat{R}|t_R], [R_1|V_R]),
\]

(128)

where \([\hat{R}|t_R]\) are elements of the space group of initial high-temperature commensurate phase, but \([R_1|V_R]\) corresponding to their transformation of four coordinate of \( \varphi \) phase. The action of operation \( \hat{g} \) on four-dimensional vector \((r, \varphi)\) can be entered as

\[
\hat{g}(r, \varphi) = (r', \varphi'),
\]

(129)

where \( r' = \hat{R}r + t_R, \varphi' = R_1\varphi + V_R \) but \( R_1 = +1 \) if \( \hat{R}q' = q' \) and \( R_1 = -1 \) if \( \hat{R}q' = -q' \).

The translation subgroup contains the following elements \([E|n], [1|q'n]\].

4.2 Bethe-Salpeter equation of incommensurate phase

In the initial hight temperature commensurate phase \( (D_{nk}^a) = 0, \langle u_{nk}^a \rangle = 0 \); the matrices of dipole-dipole interactions as well as Fourth constants transform with respect the space group symmetry this phase [16, 113, 15, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124]:

for the element

\[
\psi_{\alpha\beta}(ml, m'l') = \sum_{\nu\nu'} R_{\nu\nu'} \psi_{\nu\nu'}(nk, n'k'),
\]

(130)

where \( m + r_k = R(n + r_k) + t_R, m' + r_{k'} = R(n' + r_{k'}) + t_R \),

for translation subgroup

\[
\psi_{\alpha\beta}(nk, n'k') = \psi_{\alpha\beta}(n - m, k; n' - m, k').
\]

(131)

Correlation functions which were build on a ionic displacements as well as electronic dipole momenta operators [125, 126, 127, 128, 30, 129, 130, 131] satisfy the similar found expressions.
In low-temperature commensurate phase the non-zero averages $\langle D_n^a \rangle = d_n^a = d_{n+n_0,k}^a$ and $\langle u_n^a \rangle = C_n^a = C_{n+n_0,k}^a$ have been created, where $n_0$ is lattice vector of low temperature phase but the renormalized Fourth constants $\tilde{\psi}_{\alpha\beta}(nk, n'k')$ Eq. (120) in consideration of lattice anharmonicity transform with respect to the space group of symmetry of this phase.

The structure of the incommensurate phase is given by the average displacements and average dipole momenta which can be presented in the form of modulation waves [4]

$$\langle u_n^a \rangle = C_n^a (q' n + \varphi) = \sum_{k} C_k^a (h) \exp (i (h + q' (n + r)')),$$

where $q'$ is the wave vector of the incommensurate structure, $C_k^a (\varphi)$ and $d_k^a (\varphi)$ are periodic functions of phase $\varphi$, $h$ is an integer.

The renormalized Fourth constants $\tilde{\psi}_{\alpha\beta}(nk, n'k')$ Eq. (120) and crystal fields of lattice Eq. (112) enclose the following components

$$\tilde{\psi} = \psi + \psi^{(1)} + (\psi^{(2)} + \tilde{\psi}^{(2)}) + \ldots,$$

$$F = F^{(1)} + (F^{(2)} + \tilde{F}^{(2)}) + \ldots,$$

which

$$\psi^{(1)} = -\sigma_{DDu} (u), \quad \psi^{(2)} = -\frac{1}{2} \theta (u) (u),$$

$$F^{(1)} = \psi (D) + C (u), \quad F^{(2)} = -\sigma_{DDu} (D) (u) - \frac{1}{2} \sigma_{Duu} (u) (u),$$

the contributions from equilibrium displacements,

$$\psi^{(2)} = -\frac{1}{2} \tilde{\theta} (\tilde{u} \tilde{u}), \quad F^{(2)} = -\sigma_{DDu} (\tilde{D} \tilde{u}) - \frac{1}{2} \sigma_{Duu} (\tilde{u} \tilde{u}),$$

the contributions which connected from correlation displacements in incommensurate phase.

The crystal field of lattice Eq. (135) as well as the matrix elements Eq. (117) and the energy levels $\lambda_{nhv}$ defined by Eq. (114) in incommensurate phase vary from cell to cell. A distant from the edge of transparent region and for weak internal self-consistent crystal field of lattice the polarizability of ion (ionic group) Eq. (122) can expand in series with respect to power of intrinsic crystal fields of lattice

$$Z_{nk}^{\alpha\beta}(\omega) = Z_{0,k}^{\alpha\beta}(\omega) + \sum_{\gamma} \xi_{1,k}^{\alpha\beta,\gamma}(\omega) F_{nk}^{\gamma} + \frac{1}{2} \sum_{\gamma,\delta} \xi_{2,k}^{\alpha\beta,\gamma\delta}(\omega) F_{nk}^{\gamma} F_{nk}^{\delta} + \ldots,$$

where

$$Z_{0,k}^{\alpha\beta}(\omega) = \sum_{\alpha'} \frac{\mu_{\alpha'}^{\alpha} \mu_{\beta'}^{\beta}}{\epsilon_{\alpha'} - \epsilon_{\beta'} + \lambda_{\gamma'}} (X_{nk}^{ss} - X_{nk}^{s'}).$$

Let us known Eq. (138) in the form of sum of contributions caused different components of field $F_{nk}^{\alpha\beta}$ Eq. (135),

$$Z_{nk}^{\alpha\beta}(\omega) = \hat{Z}_{nk}^{\alpha\beta}(\omega) + Z_{1,nk}^{\alpha\beta}(\omega) + Z_{2,nk}^{\alpha\beta}(\omega) + \hat{Z}_{2,nk}^{\alpha\beta}(\omega),$$

$$Z_{1,nk}^{\alpha\beta}(\omega) = \sum_{\gamma} \xi_{1,k}^{\alpha\beta,\gamma}(\omega) F_{nk}^{(1)\gamma},$$

$$Z_{2,nk}^{\alpha\beta}(\omega) = \sum_{\gamma} \xi_{2,k}^{\alpha\beta,\gamma\delta}(\omega) F_{nk}^{(2)\gamma} F_{nk}^{(1)\delta} + \frac{1}{2} \sum_{\gamma,\delta} \xi_{2,k}^{\alpha\beta,\gamma\delta}(\omega) F_{nk}^{(1)\gamma} F_{nk}^{(1)\delta} + \hat{Z}_{2,nk}^{\alpha\beta}(\omega),$$

On the basis of Eq. (134) and Eq. (140) the Green function which satisfy of equation Eq. (121) we have presented in the form corresponding series

$$\langle \langle D | D \rangle \rangle_\omega = \langle \langle D | D \rangle \rangle_\omega^{(0)} + \langle \langle G^{(1)}(\omega) + (G^{(1)}(\omega) + \tilde{G}^{(2)}(\omega)),$$

where Green function of high temperature commensurate phase $\langle \langle D | D \rangle \rangle_\omega^{(0)}$ satisfy the equation...
\[ \langle |D| D \rangle^{(0)}_\omega = \frac{1}{\pi^2} Z_\omega - Z_0(\omega) \psi \langle |D| D \rangle^{(0)}_\omega, \]  
(145)

but the remainder contributions in Eq. (144) we have found as

\[ G^{(1)}(\omega) = 2\pi \langle |D| D \rangle^{(0)}_\omega (Z_0^{-1} Z_1 Z_0^{-1} - \psi_1) \langle |D| D \rangle^{(0)}_\omega, \]  
(146)

\[ G^{(2)}(\omega) = 2\pi \langle |D| D \rangle^{(0)}_\omega (Z_0^{-1} Z_2 Z_0^{-1} - \psi_2 - Z_0^{-1} Z_1 \psi_1) \langle |D| D \rangle^{(0)}_\omega - 2\pi \langle |D| D \rangle^{(0)}_\omega (Z_0^{-1} Z_1 \psi + \psi_1) G^{(1)}(\omega), \]  
(147)

\[ \tilde{G}^{(2)}(\omega) = 2\pi \langle |D| D \rangle^{(0)}_\omega (Z_0^{-1} Z_2 Z_0^{-1} - \tilde{\psi}_2) \langle |D| D \rangle^{(0)}_\omega, \]  
(148)

where Eq. (146), Eq. (147) were corrections linear and quadratic \((u)\) and \((D)\), Eq. (148) is correction which was proportional correlation functions \((\tilde{u}\tilde{u})\) and \((\tilde{D}\tilde{u})\).

### 4.3 Symmetry of \(\varepsilon_{ab}(\omega, k = 0)\) tensor in Incommensurate phases

We have presented \(\varepsilon_{ab}(\omega, k = 0)\) on the basis Eq. (105) and Eq. (108) in the form

\[ \varepsilon_{ab}(\omega, k = 0) = \delta_{ab} - \frac{8\pi^2}{V_0} e^2 \sum_{kk} \langle \Phi_{ka,k'p}(\omega, k = 0) \rangle, \]  
(149)

where \(\Phi_{ka,k'p}(\omega, k = 0)\) is Fourier transform of dipole-dipole Green function which includes the contributions corresponding Eq. (??)

\[ \Phi_{ka,k'p}(\omega, k = 0) = \Phi_{ka,k'p}(\omega, k = 0) + \Phi^{(1)}_{ka,k'p}(\omega, k = 0) + (\Phi^{(2)}_{ka,k'p}(\omega, k = 0) + \Phi^{(3)}_{ka,k'p}(\omega, k = 0)). \]  
(150)

First term enters into the expression for the dielectric permeability tensor for the high temperature phase

\[ \varepsilon_{0ab}(\omega, k = 0) = \delta_{ab} - \frac{8\pi^2}{V_0} e^2 \sum_{kk} \langle \Phi_{ka,k'p}(\omega, k = 0) \rangle. \]  
(151)

\[ \Phi_{0,ka,k'p}(\omega, k = 0) \] can be found from Eq. (145).

\[ \Phi_{0,ka,k'p}(\omega, k) = \frac{1}{2\pi} \delta_{kk} \psi_{0,k}(\omega) - \sum_{i} Z_{0,i}^{\alpha} \sum_{j,\beta} \psi_{k,j,\beta}(\omega) \Phi_{0,\beta,i,k'p}(\omega, k), \]  
(152)

\[ \psi_{k,j,\beta}(\omega) = \sum_{n,n'} \psi_{j,\beta}(nk, n'k') \exp^{-ik(n-n')} . \]  
(153)

The symmetry of contribution \(\Phi_{0}(\omega, k)\) is defined by space group of symmetry of high temperature commensurate phase.

In the Incommensurate phase the average displacements which include coherent phonon oscillations [132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 144, 145] were defined by modulation waves Eqs. (132), (133) as well as an internal self-consistent crystal fields of lattice Eqs. (136), (137) and the polarizabilities of ions Eqs. (141), (142), (143) were modulated with incommensurate period too

\[ F_{nk}^{\alpha} = F_{nk}(q^n + \phi), \]  
\[ Z_{nk}^{\alpha}(\omega) = Z_{nk}^{\alpha}(\omega, q^n + \phi). \]  
(154)

The renormalized elements of matrix interactions \(\tilde{\psi}\) as well as depended from \(\phi\) phase were transformed with respect into superspace group of symmetry:

for the \(((R(t), (R, (V)))\) element we have

\[ \tilde{\psi}_{ab}(ml, m'l', \phi') = \sum_{v,v'} R_{ax} R_{by} \psi_{v,v'}(nk, n'k', \phi), \]  
(155)

where

\[ m + r_l = \tilde{R}(nr_l) + t_l, \]  
\[ m' + r_l = \tilde{R}(nr_l) + t_l, \]  
\[ \phi' = R_l \phi + V_l, \]  
(156)

but for the translation subgroup \(((E| - m), (1|q| m)))

\[ \tilde{\psi}_{ab}(nk, n'k', \phi) = \psi_{ab}(n - m, k; n' - m, k'; q'| m + \phi). \]  
(157)

The dipole-dipole Green function Eq. (145) is depended from \(\phi\) phase and transforms with respect into superspace group of symmetry.
for \((\langle R|t_R\rangle, \langle R_j|V_g\rangle)\) element
\[
\langle D^β|D^β_m \rangle_{ω, ϕ} = \sum_{v v'} R_{α v} R_{β v'} \langle D^γ|D^γ_{n k'} \rangle_{ω, ϕ},
\]
(158)
but for translations subgroup
\[
\langle D^α|D^β_{n k'} \rangle_{ω, ϕ} = \langle D^α|D^β_{n-m, k'} \rangle_{ω, ϕ+q'm}.
\]
(159)
For the defining of non-zero components of gyrotropy birefringence tensor as well as gyration tensor the Fourier transformations of Green functions Eq. (159) have been necessary found.
\[
\frac{1}{N} \sum_{nn'} \langle D^α|D^β_{n k'} \rangle_{ω, ϕ} \exp^{i(k'n-kn)},
\]
(160)
where sum with respect into \(n\) and \(n'\) calculates \(N\) elements of unit cells. Hence we have found (view of subsection of Appendix)
\[
\frac{1}{N} \sum_{nn'} \langle D^α|D^β_{n k'} \rangle_{ω, ϕ} \exp^{i(k'n-kn)} = \frac{1}{N} \sum_{nn'} \langle D^α|D^β_{n k'} \rangle_{ω, ϕ+q'n} \exp^{i(k'n+(k-k)n)} = \frac{1}{N} \sum_{n} \Phi_{k a k'}(ω, k; h) \exp^{i(0) \Delta(k' - k + hq')} = \frac{1}{2π} \int_0^{2π} d ϕ' \langle \sum_{nn'} \langle D^α|D^β_{0 k'} \rangle_{ω, ϕ'} \exp^{i kn'} \rangle,
\]
(161)
where
\[
\Phi_{k a k'}(ω, k; h) = \frac{1}{2π} \int_0^{2π} d ϕ' \langle \sum_{nn'} \langle D^α|D^β_{0 k'} \rangle_{ω, ϕ'} \exp^{i kn'} \rangle.
\]
(162)
At \(k = k' \neq 0\)
\[
\frac{1}{N} \sum_{nn'} \langle D^α|D^β_{n k'} \rangle_{ω, ϕ} \exp^{i kn-n} = \Phi_{k a k'}(ω, k) = \frac{1}{2π} \int_0^{2π} d ϕ' \langle \sum_{nn'} \langle D^α|D^β_{0 k'} \rangle_{ω, ϕ'} \exp^{i kn} \rangle,
\]
(163)
but at \(k = k' = 0\)
\[
\frac{1}{N} \sum_{nn'} \langle D^α|D^β_{n k'} \rangle_{ω, ϕ} \exp^{i kn-n} = \Phi_{k a k'}(ω, k) = \frac{1}{2π} \int_0^{2π} d ϕ' \langle \sum_{nn'} \langle D^α|D^β_{0 k'} \rangle_{ω, ϕ'} \rangle.
\]
(164)
The Fourier transforms of the expressions Eqs. (163), (164) are independent from \(ϕ\) phase and can be averaged with respect to \(ϕ\) concerning of period.
We have found Fourier transformations Eqs. (161) for Eqs. (146), (147), (148)
\[
Φ^{(0)}(ω, k, h) = 2πΦ^{(0)}(ω, k + hq')\left(Z_0^{-1}(ω)Z_1(ω)hZ_0^{-1}(ω) - ψ^{(1)}(k, h)\right)Φ^{(0)}(ω, k),
\]
(165)
\[
Φ^{(2)}(ω, k, h) = 2π\Phi^{(0)}(ω, k + hq')\left(Z_0^{-1}(ω)Z_1(ω)hZ_0^{-1}(ω) - ψ^{(2)}(k, h) - Z_0^{-1}(ω)\right)\sum_h Z_1(ω)h_1(ω)h_1\Phi^{(0)}(ω, k) - 2π\Phi^{(0)}(ω, k + hq')\sum_h\left(Z_0^{-1}(ω)Z_1(ω)h - h_1\right)ψ^{(1)}(k + h, q') + ψ^{(1)}(k + h_1 q', h - h_1)\right)\Phi^{(1)}(ω, k, h_1),
\]
(166)
\[
ψ^{(j)}(k, h) = 2π\Phi^{(0)}(ω, k + hq')\left(Z_0^{-1}(ω)Z_2(ω)hZ_0^{-1}(ω) - ψ^{(2)}(k, h)\right)Φ^{(0)}(ω, k),
\]
(167)
where
\[
Z_α^β_{j,k}(ω)h = \frac{1}{2π} \int_0^{2π} d ϕ Z_α^β_{j,k}(ω, ϕ) \exp^{i(hϕ)},
\]
(168)
\[
ψ^{(j)}_{k, h}(k, h) = \frac{1}{2π} \int_0^{2π} d ϕ \langle \sum_{m} ψ^{(j)}_{l, m}(0k, ml, ϕ) \exp^{i(km)} \rangle \exp^{i(hϕ)}.
\]
(169)
The presence of a modulation wave display itself in the ionic contributions \(Z_j\) conditioned by nonlinear polarizabilities in the crystalline contributions conditioned by anharmonicities and in the mixed contributions which were proportional to \(Z_1ψ^{(1)}(k, h)\).

Hence the symmetry of the tensor \(ε_{αβ}(ω, k) = 0\) in the incommensurate phase is the same as in the initial high temperature phase. This result is correct in the presence of external field (electric, magnetic and mechanical strains).
4.4 Symmetry of $K_2SeO_4$ Incommensurate phases crystals

Let us known the symmetry of $K_2SeO_4$ crystal. Above $T_i$ temperature the symmetry these crystals was defined by $D_{sh}^{16}$-Pnam space group. The phase transition from high temperature commensurate phase in incommensurate phase conditioned the breaking of crystal equilibrium in $T_i$ point and Brillouin zone point defined by wave vector $q^I = \frac{1}{N}(1 - \delta)\mathbf{a}^*$ (where $N = 2$ for $(NH_4)_2BeF_4$ and $(N(CH_3)_4)_2MnCl_4$; $N = 3$ for $K_2SeO_4$, $Rb_2ZnCl_4$ [4]) but coherent phonon oscillations amplitude $Q(q^I, \Sigma_2)$ with defined symmetry of phonon oscillations $\Sigma_2$ was main order parameter of phase transition.

Secondary of order parameters which satisfied Eq. (126) are [4]

\[Q(q_1, \Sigma_1) \rightarrow Q^2(q_1, \Sigma_2), \quad q_1 = 2q',\]
\[Q(q_2, \Sigma_2) \rightarrow Q^2(q_2, \Sigma_2), \quad q_2 = 3q',\]
\[Q(q_3, \Sigma_3) \rightarrow Q^3(q_3, \Sigma_2), \quad q_3 = 3q' - a^*,\]
\[Q(q_4, \Sigma_4) \rightarrow Q^2(q_4, \Sigma_2), \quad q_4 = 2q' - a^*.\]

(170)

The eigenvectors have next components [4] for translation ionic displacements

\[e_k^{(t)}(q, \Sigma_1) = (e_k^{(t)}(q, \Sigma_1), (e_k^{(t)}(q, \Sigma_1), 0),\]
\[e_k^{(t)}(q, \Sigma_2) = (0, 0, (e_k^{(t)}(q, \Sigma_2)),\]
\[e_k^{(t)}(q, \Sigma_3) = (0, 0, (e_k^{(t)}(q, \Sigma_3)),\]
\[e_k^{(t)}(q, \Sigma_4) = (e_k^{(t)}(q, \Sigma_4), 0),\]

(171)

as well as for rotation motions of ionic group

\[e_k^{(r)}(q, \Sigma_1) = (e_k^{(r)}(q, \Sigma_1), (e_k^{(r)}(q, \Sigma_1), 0),\]
\[e_k^{(r)}(q, \Sigma_2) = (0, 0, (e_k^{(r)}(q, \Sigma_2)),\]
\[e_k^{(r)}(q, \Sigma_3) = (0, 0, (e_k^{(r)}(q, \Sigma_3)),\]
\[e_k^{(r)}(q, \Sigma_4) = (e_k^{(r)}(q, \Sigma_4), 0).\]

(172)

Hence the initial self-consistent crystal field $F_{nk}^a$ have in incommensurate phase the same symmetry like symmetry of translation displacement of ions.

Hence for $K_2SeO_4$ the non-zero components were $\epsilon_{12}$ gyrotropy birefringence tensor and $g_{33}$ gyration tensor specifically from above found results Eqs. (171), (172) but the numerical magnitudes these tensors we have found from experimental data [2].

It is known [4] that temperature law of order parameter $Q(q^I, \Sigma_2)$ in $K_2SeO_4$ has good agreement with $n = 2$ rank of matrix of Landau-Ginsburg-Vilson Hamiltonian for which

\[Q(q^I, \Sigma_2) \sim (T_i - T)^{\beta},\]

(173)

has been found, where $T_i$-Curie temperature, where $\beta = 0.35$. But the experimental results of gyrotropy birefringence in incommensurate phase

\[\Delta n \sim (T_i - T)^{2\beta},\]

(174)

have been predicted, where $2\beta = 0.75$ for $K_2SeO_4$ as well as $2\beta = 0.72$ for $Rb_2ZnCl_4$ defined from $\phi_{k,n,k'}^{(2)}(\omega, k = 0, h = 0)$ contribution.

4.5 Appendix

The theoretical justification of transformations Eqs. (161), (162) one can find in the subsection for $q^I$ incommensurate wave vector and arbitrary integrable concerning the Riemannian periodic function $f(\varphi) = f(\varphi + 2\pi)$ in the limit $N \rightarrow \infty$ we have found

\[\frac{1}{N} \sum_n f(\varphi + q' n) \exp(nk) = \sum_h \left(\frac{1}{2\pi} \int_0^{2\pi} f(\varphi') \exp(ih\varphi') d\varphi' \right) \exp(ih\varphi) \Delta(k + hq^I),\]

(175)

where

\[\Delta(k) = \sum_{k_0, k \neq k_0} \delta G,\]

(176)

$G$ is inverse lattice vector.

Let us known the $d = 1$ one-dimensional modulation. For the incommensurate of $q^I = \delta a^*$ wave vector the order parameter $\delta$ can be presented in the form $\delta = \frac{N_1}{N_0}$, where $N_1$ and $N_0$ are integers which have no common divisor and
lim_{N_0 \to \infty} \frac{N}{N_0} = \delta. Let us known in crystal the main region which includes $N_0$ cells along $\mathbf{a}$ and we have imposed the Born von Karman periodic boundary conditions. Then for wave vector $\mathbf{k}$ we have

$$\mathbf{k} = \frac{n}{N_0} \mathbf{a}^* \quad (m = 0, 1, \ldots, N_0 - 1), \quad (177)$$

but for the vector of incommensurate phase

$$\mathbf{q}^f = \frac{N}{N_0} \mathbf{a}^*. \quad (178)$$

The expression Eq. (175) if based on Eqs. (177), (178) can be rewritten in the form

$$\frac{1}{N_0} \sum_{n=0}^{N_0-1} f(\varphi + 2\pi \frac{N}{N_0} n) \exp(2i\pi \frac{m}{a} n) = \frac{1}{N_0} \sum_{n=0}^{N_0-1} f(\varphi + 2\pi \frac{N}{N_0} n) \exp(2i\pi \frac{m}{a} n), \quad (179)$$

where at $[\ldots]$ has been accounted fractional part of number.

We have found that for $0 \leq n \leq N_0 - 1$ the sets $\mathbf{n} \frac{N}{N_0}$ and $\{ n \mathbf{N} \}$ are equivalents. The study sets include $N_0$ elements from the interval $[0, 1]$ and the study sets can be like to single-valued correspondence as well as the study sets can not be like to single-valued correspondence. In the second case one can find the such $n_1 \neq n_2$ that

$$\{ n \mathbf{N} \} \neq \{ m \mathbf{N} \}. \quad (180)$$

Then

$$\frac{N_1}{N_0} n_1 - \frac{N_0}{N_0} n_2 = M, \quad (M \in \mathbb{Z}), \quad (181)$$

or $N_1(n_1 - n_2) = MN_0$. If $N_1$ and $N_0$ have no common divisor we have found

$$n_1 - n_2 = MN_0, \quad (M \in \mathbb{Z}). \quad (182)$$

At $n_1 \geq 0$, $N_0 - 1 \geq n_2$ the latter expression is true only for $M_1 = 0$ i.e. $n_1 = n_2$ and so we have found contradiction. Hence we have found that the first case is true and the formula Eq. (179) can be rewritten in the form

$$\sum_{-\frac{N_0}{2} < h \leq \frac{N_0}{2}} \left( \frac{1}{N_0} \sum_{n=0}^{N_0-1} f(\varphi + 2\pi \frac{N}{N_0} n) \exp(-i\hbar \frac{N}{N_0} n) \Delta\left( \frac{m+h\mathbf{N}}{N_0} \mathbf{a}^* \right) \right) = \sum_{-\frac{N_0}{2} < h \leq \frac{N_0}{2}} \frac{1}{2N_0} \sum_{n=0}^{N_0-1} \left( f(\varphi + 2\pi \frac{N}{N_0} n) \exp(-i\hbar \varphi \frac{N}{N_0} + \frac{2\pi}{a} n) \right) \exp(i\frac{h}{\hbar} \Delta(k + h\mathbf{q}^f)), \quad (183)$$

where we impose the identity

$$\exp(i\frac{2\pi}{a} n) = \sum_{-\frac{N_0}{2} < h \leq \frac{N_0}{2}} \exp(-2i\pi \frac{h}{N_0} n) \Delta\left( \frac{m+h\mathbf{N}}{N_0} \mathbf{a}^* \right). \quad (184)$$

We have entered in Eq. (183) the following key $\Delta \varphi = \frac{2\pi}{N_0}$, $\varphi = \frac{2\pi}{N_0} n = n\Delta \varphi$ we have found the Riemann integral in Eq. (183) in limit $N_0 \to \infty$. Hence Eq. (175) is true.

## 5 Quantized solutions of one-dimensional Maxwell equations for thin ferroelectric films in Incommensurate phase with space dispersion in framework of Landau-Ginsburg-Devonshire theory

The structure of incommensurate phase is given by the average displacements as well as average dipole momenta and material parameters which may be presented in the form of modulation waves

$$(u^a_{n_0}) = c^a_{\varphi}(\mathbf{q}^f \mathbf{n} + \varphi) = \sum_{h} c^a_{\varphi}(h) \exp(i q^f (n + \varphi) + \varphi), \quad (185)$$

$$(D^a_{n_0}) = d^a_{\varphi}(\mathbf{q}^f \mathbf{n} + \varphi) = \sum_{h} d^a_{\varphi}(h) \exp(i q^f (n + \varphi) + \varphi), \quad (186)$$

where $\mathbf{q}^f$ is the wave vector of the incommensurate structure, $c^a_{\varphi}(\varphi)$ as well as $d^a_{\varphi}(\varphi)$ are periodic functions of phase $\varphi$ with $2\pi$ period, $h$ is an integer.

From Maxwell equations

$$\text{rot} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (187)$$

if we have assumed that the magnetic induction $\mathbf{B}$ in Maxwell equation was independent quantity with respect to time then
rot\mathbf{E} \approx 0. \tag{188}

Let us known for the vector of electric field strength following expressions

\[ \mathbf{E} = -\nabla \varphi, \tag{189} \]

as well as for the vector of electric-flux density the following Maxwell equations

\[ \text{div}\mathbf{D} = \text{div}(\mathbf{E} + 4\pi\mathbf{P}) = \frac{4\pi}{\varepsilon_0} \rho(\varphi), \tag{190} \]

where \( \rho(\varphi) \) is charges density, \( \varepsilon_0 \) is dielectric constant.

The spontaneous polarization \( P_3 \) is directed along the polar axis \( z \). Further we assume that the dependence of polarization vector acquires the form [137, 138, 139]

\[ \mathbf{P}(\mathbf{r}) = [\varepsilon_0(\varepsilon_{11} - 1)E_1, \varepsilon_0(\varepsilon_{11} - 1)E_2, P_3(\mathbf{r}, \mathbf{E}) + \varepsilon_0(\varepsilon_{33}^b - 1)E_3]. \tag{191} \]

Let us known the quasiequilibrium polarization distribution \( P_3(x, y, z) \) with taking into account the gyrotropy effects which are shown to be found from Euler-Lagrange boundary problem in the framework of Landau-Ginsburg-Devonshire theory [137, 138, 139] as follows

\[ \alpha P_3 + \beta |P_3|^2 P_3 - g(\Delta_\perp + \frac{\partial^2 P_3}{\partial z^2})P_3 = -\frac{\partial \varphi}{\partial z}, \tag{192} \]

with the next boundary conditions

\[ (P_3 + \lambda_1 \frac{\partial P_3}{\partial z})|_{z=0} = -P_b, \tag{193} \]

\[ (P_3 - \lambda_2 \frac{\partial P_3}{\partial z})|_{z=-L} = 0, \tag{194} \]

with taking into consideration for the vector of electric field strength the following expressions \( E_z = -\frac{\partial \varphi}{\partial z} \).

We seek the solution the Euler-Lagrange equation Eq. (192) in the form

\[ P_3(x, y, z) = P_3 \exp(ikr) \exp(i k z). \tag{195} \]

Substituting the solution Eq. (195) in Eq. (192) we have found non-linear equation with respect to \( P_3 \)

\[ \alpha P_3 + \beta P_3^3 + g k_1^2 P_3 - g \frac{\partial^2 P_3}{\partial z^2} P_3 = -\frac{\partial \varphi}{\partial z}, \tag{196} \]

Hence we have found the next non-linear cubic equation with respect to polarization vector component \( P_3 \)

\[ f = \alpha P_3 + \beta P_3^3 + g k_1^2 P_3 + g k_2^2 P_3 = 0, \tag{197} \]

the solutions of which meet the next expressions

\[ P_3^{(1)} = \frac{1}{6} \left( \frac{12\sqrt{3\varkappa+108\varepsilon_0}}{\varepsilon_0} \right) \left( \frac{f_2}{f_1^{1/3}} - \frac{2g(k_2^2 + k_1^2 + \alpha)}{(12\sqrt{3\varkappa+108\varepsilon_0})f_1^{1/3}} \right), \tag{198} \]

as well as

\[ P_3^{(2)} = P_3^{(3)} = -\frac{1}{12} \left( \frac{12\sqrt{3\varkappa+108\varepsilon_0}}{\varepsilon_0} \right) \left( \frac{f_2}{f_1^{1/3}} + \frac{(g(k_2^2 + k_1^2 + \alpha))}{(12\sqrt{3\varkappa+108\varepsilon_0})f_1^{1/3}} \right), \tag{199} \]

where

\[ R = \sqrt{\frac{1}{\beta}(4g^3(k_1^6 + k_2^6) + 12g^3k_2^2k_1^2(k_1^2 + k_2^2) + 12\alpha g^2(k_1^2 + k_2^2)^2 + 12\alpha^2 g(k_1^2 + k_2^2)^2 + 27E_2^2\beta + 4\alpha^3)}. \tag{200} \]

The Maxwell equations we have specified for three regions: for contact clearance \((-H - L < z < -L)\), for ferroelectric thin polar-active nanofilm \(-L < z < 0\) and for semiconductor \(z > 0\) [137, 138, 139]

\[ \left( \frac{\partial^2 \varphi}{\partial z^2} + \Delta_\perp \varphi \right) = 0, \quad -H - L < z < -L, \]

\[ \varepsilon_{33}^b \frac{\partial^2 \varphi}{\partial z^2} + \varepsilon_{11} \Delta_\perp \varphi = \frac{1}{\varepsilon_0} (\varepsilon_{33}^b \frac{\partial \varphi}{\partial z} - \rho_f(\varphi)), \quad -L < z < 0, \]

\[ \varepsilon_0 \varepsilon_{11} \left( \frac{\partial^2 \varphi}{\partial z^2} + \Delta_\perp \varphi \right) = -\rho_f(\varphi), \quad z > 0, \tag{201} \]

where \( \varepsilon_{33}^b, \varepsilon_{11} = \varepsilon_{22} \) are permittivities of gyrotrropic medium with taking to account the optical activity effects, \( \Delta_\perp = \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2} \).
For the light which propagates along Oz axis for first contact clearance $(-H - L < z < -L)$ the solution of Maxwell equation for the vector of electric field strength we have found by using Green function method

$$\text{div} \mathbf{E} = \frac{4\pi \rho}{\varepsilon_0},$$

(202)

where $\rho$ is charge density,

$$\frac{\delta \varepsilon \mathbf{E}}{\delta \varepsilon x} = \frac{4\pi \rho(z)}{\varepsilon_0},$$

(203)

where $\mathbf{E}$ is unit vector in Oz axis

$$\frac{\delta \varepsilon G}{\delta \varepsilon x} = \frac{4\pi \delta(z)}{\varepsilon_0},$$

(204)

where $G_\varepsilon$ is Green step Hevithaide function, $\delta(z)$ is Dirac delta function,

$$G_\varepsilon \equiv \varepsilon_2 G = \frac{4\pi \varepsilon}{\varepsilon_0} \varepsilon = \frac{4\pi z}{\varepsilon_0 |z|},$$

(205)

The solution the equation Eq. (203) we have found in the form

$$E(z_0) = \int G(z_0 - z) \rho(z) dz + C_1.$$ 

(206)

Since for scalar potential we have found the following equations

$$E = -\text{grad} \varphi = -\varepsilon_i \frac{\partial \varphi}{\partial x_i},$$

(207)

then

$$\varepsilon_i \varepsilon_j E_i = -\varepsilon_i \frac{\partial \varphi}{\partial x_i},$$

(208)

hence for the solution the Eq. (208) we have derived a formula in the form

$$\varphi = -\int E_z dz + C_2.$$ 

(209)

Substituting Eq. (206) in Eq. (209) the solution the Maxwell equation for the first contact clearance $(-H - L < z < -L)$ we have presented like follows

$$\varphi = -\int z_0 \int G(z_0 - z) \rho(z) dz + C_1 + C_2.$$ 

(210)

The one-dimensional Maxwell wave equations system of Incommensurate phase crystal thin film for light which propagates along Oz axis with space dispersion is found in the form [140, 141, 142, 143, 144, 145]

$$\nabla^2 \mathbf{E} + \frac{\omega^2}{c^2} \mathbf{D} = 0,$$

(211)

where

$$D_i(r, \varphi) = \varepsilon_{ij} E_j(r, \varphi) + \varepsilon_{ij} (q r + \varphi) \varepsilon_{ij} (q r + \varphi) \varepsilon_{ij} (q r + \varphi) \varepsilon_{ij} (q r + \varphi) \varepsilon_{ij} (q r + \varphi) \varepsilon_{ij} (q r + \varphi) + \frac{1}{2} \nabla \gamma_{ijl} (q r + \varphi) \cdot \mathbf{E}_{j}(r, \varphi),$$

(212)

$$\gamma_{ijl} = \varepsilon_{ijm} \varepsilon_{ml},$$

(213)

$$\varepsilon_{12} (q r + \varphi) = \varepsilon_{12} \cos(q r + \varphi),$$

(214)

$$\varepsilon_{23} (q r + \varphi) = \varepsilon_{23} \sin(q r + \varphi),$$

(215)

It is known [1] vibrations with sine-type modulations can be induced by the impulsive stimulated Raman-scattering process resulting in sine-type oscillations. It explains the coherent oscillations observed in transparent compounds under pumping with a photon energy smaller than an optical gap [1]. In opaque materials the abrupt modification of pump photons can trigger displaced motions of ions towards new coordinates in the excited state resulting in cosine-type oscillations [1].

Hence we have replaced the Eqs. (201) in the space region $-L < z < 0$ by of the one-dimensional Maxwell wave equations system of Incommensurate phase crystal thin film of the space region $-L < z < 0$ for light which propagates along Oz axis in consideration of both the Gyrotropy effect and dipoles on interface of ferroelectric thin film in incommensurate phase/semiconductor nanoheterostructures as well as free charges for specifying of charge quantization is found in the form
In the first approximation the solutions for wave vectors and field may be now written as

\[ (((-2c^2 \Delta_t + \epsilon)\delta_{ij} + e_{ij}\delta_{ij}\cos(q_i^0 x_i + \varphi) + \gamma_{ijkl} \sin(q_i^0 x_i + \varphi) \nabla_i + \frac{1}{2} \gamma_{ijkl} \cos(q_i^0 x_i + \varphi)q_i^0 E_j) = 0, \quad (216) \]

where \( x_i \equiv z, \epsilon \equiv \epsilon + 4\pi P_3 - 4\pi \int_{|z|=1} \rho_j(z) dz + C_i. \) We have studied translation symmetry breaking at the normal to interface of ferroelectric thin film in incommensurate phase/semiconductor nanoheterostructures in the direction of light propagation along \( Oz \) axis too.

The sought for solution of equation can be presented by following expression according to theorem of Phlocke-Lyapunov

\[ E_j = \psi_j(q_i^0 x_i + \varphi)e^{ik_n x_n}, \quad (217) \]

In according with theorem of Dirikhle the amplitudes of the solutions are presented by the plane waves

\[ \psi_j(q_i^0 x_i + \varphi) = \sum_{n=-\infty}^{\infty} C^{(n)} e^{in(q_i^0 x_i + \varphi)}. \quad (218) \]

The wave vectors can be extended by series

\[ k_i = k^{(0)}_i + k^{(1)}_i + k^{(2)}_i + ... \quad (219) \]

Substituting Eqs. (217), (218), (219) in Eq. (216)

\[ \sum_{n=-\infty}^{\infty} (\epsilon_{ij} + \frac{1}{2} \gamma_{ijkl} q_j^0/) = \sum_{n=-\infty}^{\infty} \frac{1}{2} \gamma_{ijkl} (k^{(0)}_i + k^{(1)}_i + k^{(2)}_i + ...) + n q_j^0 = \sum_{n=-\infty}^{\infty} \frac{1}{2} \gamma_{ijkl} (k^{(0)}_i + k^{(1)}_i + k^{(2)}_i + ...) + n q_j^0) \]

The sought for solutions of Eq. (216) one be found as two linear polarization wave in the zero approximation

\[ (k^{(0)}_i)^2 = \frac{\omega^2}{c^2} e_{ii}, \quad (221) \]

\[ C^{(0)}_{1x} = 1, \quad C^{(0)}_{1y} = 0, \quad (222) \]

\[ C^{(0)}_{2x} = 0, \quad C^{(0)}_{2y} = 1. \quad (223) \]

In the first approximation the solutions for wave vectors and field may be now written as

\[ k_i^{(1)} = 0, \quad (224) \]

\[ C^{(\pm 1)}_{2x} = \frac{\omega^2 \epsilon_{ij} \pm \sqrt{\epsilon_{ij} \pm \gamma_{ijkl} q_j^0}}{(k^{(0)}_i + k^{(1)}_i + k^{(2)}_i + ...) - (k^{(0)}_i + k^{(1)}_i + k^{(2)}_i + ...)} \]

\[ C^{(\pm 1)}_{1x} = 0, \quad C^{(\pm 1)}_{1y} = \frac{\omega^2 \epsilon_{ij} \pm \sqrt{\epsilon_{ij} \pm \gamma_{ijkl} q_j^0}}{(k^{(0)}_i + k^{(1)}_i + k^{(2)}_i + ...) - (k^{(0)}_i + k^{(1)}_i + k^{(2)}_i + ...)}. \quad (225) \]

Hence

\[ \psi_1(q_i^0) = \frac{1}{|e^{i k_i^{(0)} x_i + \varphi}|} \exp^\pm (q_i^0 x_i + \varphi) \]

\[ \psi_2(q_i^0) = \frac{1}{|e^{i k_i^{(0)} x_i + \varphi}|} \exp^\pm (q_i^0 x_i + \varphi) \]

In the second approximation the sought for solutions in series can be found from equation

\[ \frac{c^2}{\omega^2} (k^{(2)}_i)^2 + 2k^{(0)}_i k^{(2)}_i C^{(0)}_i = 0 \]

\[ = (\epsilon_{ij} + \frac{1}{2} \gamma_{ijkl} q_j^0) C^{(0)}_i - \frac{1}{2} \gamma_{ijkl} k^{(0)}_i C^{(1)}_{ij} - \frac{1}{2} \gamma_{ijkl} k^{(0)}_i C^{(2)}_{ij} = 0, \quad (229) \]

\[ \frac{c^2}{\omega^2} k^{(2)}_i = -k^{(0)}_i + \sqrt{(k^{(0)}_i)^2 \frac{c^2}{\omega^2} + 2 \frac{c^2}{\omega^2} (\epsilon_{ij} + \frac{1}{2} \gamma_{ijkl} q_j^0) C^{(0)}_i - \frac{1}{2} \gamma_{ijkl} k^{(0)}_i C^{(2)}_{ij} \frac{c^2}{\omega^2} - \frac{1}{2} \gamma_{ijkl} k^{(0)}_i C^{(2)}_{ij} \frac{c^2}{\omega^2}), \quad (230) \]
\[ \hat{k}_{ij}^{(2)} = -k_i^{(0)} - \frac{\omega^2}{c^2} \sqrt{(k_i^{(0)})^2 c_i^4 + 2 \frac{c_i^2}{\omega^2} \left( \epsilon_{ij} + \frac{1}{2} \gamma_{ij} q_i^0 \right) C_{ij}^{l+} - \gamma_{ij} k_i^{(0)} C_{ij}^{l-}}. \]  

(231)

Hence the values of natural optical gyrotropy as well as Rashba spin splitting are shown to be derived like \((k^{(0)} \pm \hat{k}^{(2)})^2\) as displacement of two symmetrically allocated parabolas from Brillouin zone center. We have found the formulas of Gyrotropy splitting of \(Rb_2ZnBr_4\) as well as \(K_2SeO_4\). Incommensurate phases crystal and the magnitude of this Gyrotropy splitting are found to be related with Plank constant as well as electron mass like \(\epsilon = \frac{k^{(0)1} k^{(2)}}{2m_{el} c^2} = 138.7202 eV/cm^2\) but the corresponding wave vector was estimated to be \(k = 7.1614 \times 10^5 cm^{-1}\). The magnitude of these displacements were estimated correspondingly with formulas Eqs. (230), (231). Hence in the paper we have derived the formulas of Gyrotropy splitting of \(Rb_2ZnBr_4\) and \(K_2SeO_4\). Incommensurate phases crystal from Maxwell wave equations and the magnitude of this Gyrotropy splitting are found to be related with light velocity like \(\epsilon = hck/eV = 14.0798 eV\) but the corresponding wave vector was estimated to be \(k = 7.1614 \times 10^5 cm^{-1}\).

Table 4. The material parameters \(Rb_2ZnBr_4\): refractive index \(n = \frac{k}{\omega}\), gyrotropic birefringence \(\Delta n_{11}\), Gyrotropy \(g\), light wavelength \(\lambda\) in nm, the frequency of light in \(c^{-1}\), lattice constant \(c^*\) in A, irrational parameter \(\gamma\), wave vector of Incommensurate phase \(q_i^l\) in \(cm^{-1}\) and temperature \(T\) in K [2], the Gyrotropy splitting \(e\) in eV, the wave vector of Gyrotropy splitting \(k\) in \(cm^{-1}\).

<table>
<thead>
<tr>
<th>(\frac{n - \frac{k}{\omega}}{0.0000 - 0.0001})</th>
<th>(\Delta n_{11})</th>
<th>(\lambda)</th>
<th>(\Omega)</th>
<th>(c^*)</th>
<th>(\gamma)</th>
<th>(q_i^l)</th>
<th>(T)</th>
<th>(e)</th>
<th>(k)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.65</td>
<td>5.3 \times 10^{-3}</td>
<td>10^{-5}, 10^{-4}</td>
<td>632.8</td>
<td>2.9788 \times 10^{15}</td>
<td>9.71</td>
<td>0.293</td>
<td>3.0175 \times 10^{6}</td>
<td>300</td>
<td>14.0798</td>
</tr>
</tbody>
</table>

The analytical solutions of the Maxwell wave equations as well as natural optical gyrotropy effects are found in \(Rb_2ZnBr_4\) as well as \(K_2SeO_4\). Incommensurate phases crystals connected with giant light velocity as well as via interaction with coherent phonon oscillations [54, 55, 56, 28, 58, 59, 60, 61, 62, 63, 64, 65, 146, 75, 147, 54, 55, 56]. In the framework of the superspace symmetry group theories the Maxwell wave equations are solved which are shown to be connected with the symmetry group of \(D_{3h}^{18}\) or isomorphic groups [58, 59, 60, 61, 62, 63, 64, 65, 146, 75, 147]. In the paper the non-zero gyration \(g_{33}\) and gyrotropic birefringence \(\epsilon_{12}\) tensors of \(K_2SeO_4\) and \(Rb_2ZnBr_4\) materials based on \(D_{3h}^{18}\) space symmetry group were found. The values of natural optical gyrotropy as well as Rashba spin splitting are shown to be found like \((k^{(0)} \pm \hat{k}^{(2)})^2\) as displacement of two symmetrically allocated parabolas from Brillouin zone center. In the article the natural optical gyrotropy effects are shown to be found with light velocity like \(\epsilon = hck/eV = 14.0798 eV\) but the corresponding wave vector was estimated to be \(k = 7.161 \times 10^5 cm^{-1}\) [148, 42, 43, 44, 75, 86]. The found strong natural optical gyrotropy has been based on available experimental data [2].

References


